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The shock-expansion method and Whitham's rule

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The assumptions of the shock-expansion method are re-examined. Although the shock-expansion method and Whitham's rule are used to treat two different classes of problems, certain similarities between these two methods are noted. It is suggested that a single procedure leads to solutions for the entire flow field for both classes of problems.

1. Introduction

Whitham (1958) has suggested a simple rule in order to determine the motion of a shock wave through regions of non-uniform area or flow. The rule is to apply the differential relation which is valid along a characteristic to the flow quantities just behind the shock wave. This relation plus the Rankine–Hugoniot conditions across the shock are then sufficient to determine the motion of the shock for a large but restricted class of problems. The accuracy of the results is extremely good. Whitham applies the results to several examples, of which a typical illustration is the propagation of a shock wave down a tube with variable cross-sectional area. This problem was studied previously by Chester (1954) and Chisnell (1957), who obtained the same results but in a more complicated manner.

No basic reason is given for the rule except that it works! But Whitham does suggest that the method is related in some way to the shock-expansion method. The shock-expansion method treats a somewhat different class of problems. Typical ones are: (1) the steady, two-dimensional supersonic flow over an airfoil and (2) the unsteady, one-dimensional propagation of a shock wave caused by a piston with variable speed.

In the present paper, the assumptions basic to the shock-expansion method are re-examined and the method of solution is stated in a slightly different manner from that previously used. This same formulation can be used to study shock propagation in a non-uniform region. It reduces to Whitham's rule at the shock front and in addition suggests a method of calculating the entire flow field. The method is only applicable when disturbances are propagated predominantly along one set of characteristics, a limitation implicit in both the shock-expansion method and Whitham's rule.

2. Shock-expansion method

Although the shock-expansion method is applicable to both steady two-dimensional flows and unsteady one-dimensional flows, only the latter will be discussed here. The same analysis can be applied to the two-dimensional case.

The basic equations for the one-dimensional time-dependent flow of an inviscid, non-conducting perfect gas are

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial \phi}{\partial t}+u\frac{\partial \phi}{\partial x} = 0, \tag{2.3}$$

where u, ρ, p, and ϕ are the velocity in the x-direction, density, pressure, and entropy, respectively. The equation of state is $p = \rho RT$, and the isentropic speed of sound is defined as $a^2 = (\partial p/\partial \rho)_\phi = \gamma RT$, where γ is the ratio of specific heats.

The above equations can be combined in the usual way to form the characteristic relations, which can be written as

$$dp+\rho a\, du = 0 \quad \text{on} \quad C_+ \text{ characteristics}, \quad dx/dt = u+a; \tag{2.4}$$

$$dp-\rho a\, du = 0 \quad \text{on} \quad C_- \text{ characteristics}, \quad dx/dt = u-a; \tag{2.5}$$

$$dp-a^2 d\rho = 0 \quad \text{on} \quad P, \text{ particle path lines}, \quad dx/dt = u. \tag{2.6}$$

An alternative way of writing these equations which is sometimes convenient is

$$dr-\frac{a}{2(\gamma-1)C_p}d\phi = 0 \quad \text{on} \quad C_+, \tag{2.7}$$

$$ds-\frac{a}{2(\gamma-1)C_p}d\phi = 0 \quad \text{on} \quad C_-, \tag{2.8}$$

$$d\phi = 0 \quad \text{on} \quad P, \tag{2.9}$$

where

$$r = \frac{a}{\gamma-1}+\tfrac{1}{2}u, \tag{2.10}$$

$$s = \frac{a}{\gamma-1}-\tfrac{1}{2}u. \tag{2.11}$$

As an example of the application of the shock-expansion method, consider the time-dependent, one-dimensional motion of a piston accelerated from rest with non-zero initial velocity (see figure 1). A shock is formed instantaneously and, because the gas is at rest in $t < 0$, the gas ahead of the shock is motionless and in a uniform state. Disturbances originating from the piston surface and due to the variable motion of the piston propagate along positive C_+ characteristics. These disturbances are partly absorbed by the shock (the speed of which is altered) and are partly reflected. The reflected waves propagate along negative C_- characteristics and eventually influence the pressure on the piston surface.

Epstein (1931) first suggested a simplified method to determine the flow field and, in particular, the pressure on the piston. In his method, the initial velocity of the shock and the properties immediately behind the shock are calculated from the usual Rankine–Hugoniot conditions across a shock. The calculations are extended to the entire flow field by assuming that the flow behind the leading-edge shock is the same as that in an isentropic Prandel–Meyer expansion. In this approximation, one characteristic parameter, say s, is constant throughout the flow field, and r and therefore p, ρ, and u are constant along the positive C_+ characteristics. Another way of saying this is that ϕ is assumed constant and then equation (2.5) (or equation (2.8)) is applied throughout the flow field and not only on a negative characteristic.

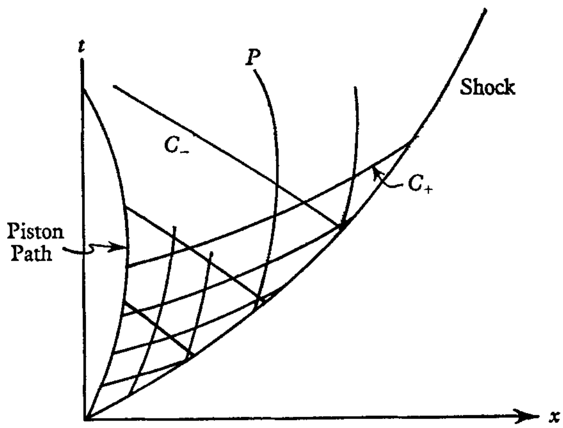

FIGURE 1. The (x, t) diagram for propagation of a shock wave caused by variable speed piston. Positive characteristics are denoted by C_+, negative characteristics by C_-, and particle path lines by P.

Eggers & Syvertson (see Eggers, Syvertson & Kraus 1953) have improved the shock-expansion method by taking into account an approximation for the non-isentropic character of the flow field. Their method may be restated as follows: Retain the characteristic equations valid along particle path lines P and along positive characteristics C_+, equations (2.4) and (2.6). The conditions at the piston surface may be calculated by using equation (2.6) and by assuming that equation (2.5) is valid for a fluid particle at the piston surface. The rest of the flow field can be calculated by using equation (2.4) and by assuming that equation (2.5) is valid on positive characteristics. From this it can be seen that p and u are constant along positive characteristics, although entropy and other variables are not. The use of both approximations (p and u constant along C_+) at the shock gives slightly different results. Either one or the other (or the average (Eggers *et al.*)) can be used.

The results of this method are extremely good and the time required is quite small by comparison with that needed for the solution by the method of characteristics.

The neglect of the C_- relations implies the neglect of reflexions. However, as can be seen above, the assumption used to replace this relation is not unique. Still other assumptions are possible, as will be seen below.

The excellent results obtained by the shock-expansion method imply that the basic approximation is valid, i.e. equation (2.5) is approximately valid along a particle path line at the piston surface and also approximately valid along positive characteristics. Since any particle path line can be interpreted as a possible piston curve, it follows that equation (2.5) is approximately true along any particle path line. This result is substantiated by the numerical calculations of Mahoney (1955).

Indeed, another way of determining the flow field, equivalent in accuracy to the shock-expansion method, is to use equations (2.4) and (2.6) and apply equation (2.5) along a streamline (or use equations (2.7) and (2.9) and apply equation (2.8) along a streamline). The variation of s along the shock is needed and can be obtained from the Rankine–Hugoniot shock conditions. This relation can also be replaced by equation (2.8). This approximation was first noticed and used by Pillow (1949), but in a different manner. The approximation involved can be seen readily for a weak shock by comparing $ds/d\phi$ along a shock and $ds/d\phi$ as given by equation (2·8). For a weak shock with velocity U and Mach number $M = U/a_0$, where $\epsilon = M^2-1 \ll 1$, the shock conditions give

$$\frac{s-s_0}{a_0} = \frac{\epsilon^3}{8(\gamma+1)}+O(\epsilon^4), \tag{2.12}$$

$$\frac{\phi-\phi_0}{C_v} = \frac{2\gamma(\gamma-1)}{3(\gamma+1)^2}\epsilon^3+O(\epsilon^4). \tag{2.13}$$

From equation (2.12), (2.13) and (2.8) it follows that

$$\left(\frac{ds}{d\phi}\right)_{C_-}\bigg/\left(\frac{ds}{d\phi}\right)_{\text{shock}} = \frac{8}{3\gamma(\gamma-1)} \tag{2.14}$$

to a first approximation. For $\gamma = \frac{5}{3}$, this is exactly one, but the approximation is less accurate as $\gamma \to 1$.

Since equation (2.5) (and equation (2.8)) is approximately valid along a particle path line and along a positive characteristic, it is approximately true throughout the flow field.

3. Whitham's rule

Since the shock-expansion method gives such good results, it seems natural to attempt other problems using the same procedure. To illustrate the extension, consider a shock propagating down a tube of variable area $A(x)$ with constant flow conditions ahead of the shock. This problem has been studied previously by Whitham (1958), Chester (1960), and Chisnell (1957). It is assumed that $A(x)$ is constant for $x < 0$ (see figure 2) and that the shock moves with constant velocity in this region. For $x > 0$ the area of the tube changes with distance, as must the shock velocity. As the shock velocity changes, disturbances propagate along the negative characteristics and interact (1) with the entropy changes across particle path lines (contact discontinuities), and (2) with area changes. The contact discontinuities also interact with the area changes. These interactions cause waves which then interact with the shock.

The appropriate equations of motion are

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\rho u}{A}\frac{dA}{dx} = 0, \quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \quad \frac{\partial \phi}{\partial t}+u\frac{\partial \phi}{\partial x} = 0. \qquad (3.1), (3.2), (3.3)$$

The characteristic relations can be written as

$$dp+\rho a\,du+\{\rho a^2u/(u+a)\}(dA/A) = 0 \quad \text{on} \quad C_+, \quad dx/dt = u+a, \tag{3.4}$$

$$dp-\rho a\,du+\{\rho a^2u/(u-a)\}(dA/A) = 0 \quad \text{on} \quad C_-, \quad dx/dt = u-a, \tag{3.5}$$

$$dp-a^2\,d\rho = 0 \quad \text{on} \quad P, \quad dx/dt = u. \tag{3.6}$$

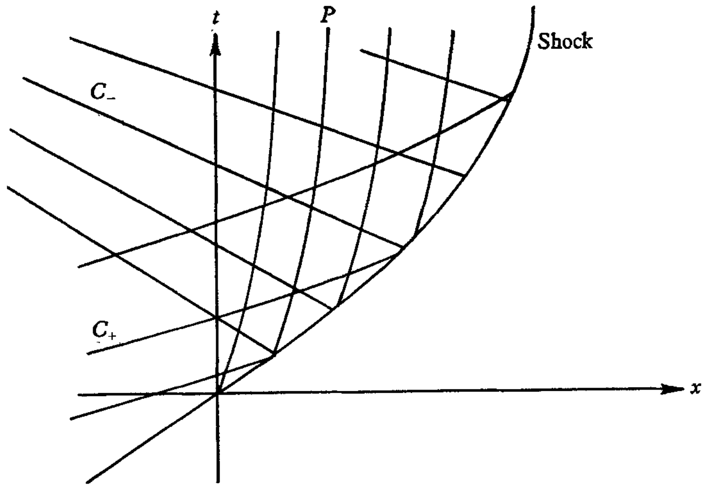

FIGURE 2. The (x, t) diagram for the propagation of a shock wave into a tube of variable area.

In this problem, the major disturbances are transmitted along the negative characteristics and the positive characteristics carry relatively little information. To extend the shock-expansion method as stated above, equations (3.5) and (3.6) are used and it is assumed that equation (3.4) is approximately true throughout the flow. The motion of the shock can be found by applying equation (3.4) along the shock and using the Rankine–Hugoniot shock conditions. This is just Whitham's rule. It has been verified that this result is extremely accurate for (1) weak shocks (Chester), and (2) strong shocks in a converging channel (Guderley 1942).

In addition, the entire flow field can be calculated by this method. Comparison with the exact method of characteristics has not yet been made. However, a limited check of the assumption that equation (3.4) is approximately true throughout the flow can be obtained from the linearized theory (first investigated by Chester). Upon linearization, equation (3.4) becomes

$$dp + \rho_1 a_1 du + \{\rho_1 a_1^2 u_1/(u_1 + a_1)\}(dA/A_1) = 0 \qquad (3.7)$$

on each C_+. But each C_+ characteristic starts in a uniform region, where $p = p_1$, $\rho = \rho_1$, etc., and therefore, upon integration, the above equation becomes

$$p - p_1 + \rho_1 a_1(u - u_1) + \{\rho_1 a_1^2 u_1/(u_1 + a_1)\}(A - A_1)/A_1 = 0. \qquad (3.8)$$

Upon differentiation, this equation is identical in form with equation (3.7), but is now valid throughout the flow. Therefore, within the linearized approximation, this confirms the original conjecture.

4. Conclusion

From these foregoing arguments, it is suggested that a single procedure is sufficient to solve both classes of problems mentioned above. This procedure is to use the characteristic relations valid along the particle path lines and the principal characteristics. The third relationship that is required is obtained from the statement that the equation which is *strictly* valid only along a minor characteristic is *approximately* true throughout the flow field.

The limitations to the method can be seen from the following: The equation along a characteristic gives information on disturbances travelling along the characteristic. When the minor characteristic equation is replaced, as suggested above, the disturbances travelling along the minor characteristics can no longer be described adequately. In particular, reflexions of the major disturbances, which travel along minor characteristics, and any disturbances originating at the boundaries and travelling along minor characteristics are neglected. Disturbances propagating along major characteristics are accounted for properly.

In the variable-area problem, the procedure is not applicable, for instance, if (1) disturbances are caused at $x = 0$ by a piston with variable speed, or if (2) the disturbances propagating along negative characteristics modify the disturbances propagating along positive characteristics to an appreciable extent. A particular example for which the present method is invalid is the Taylor blast wave problem, i.e. a strong shock wave propagating into an expanding channel. For a spherical shock wave caused by a finite amount of energy released instantaneously at a point, Taylor (1950) shows that the shock velocity U is proportional to $R^{-\frac{3}{2}}$, where R is the distance from the origin. Whitham's rule gives U proportional to $R^{-0\cdot394}$ (for $\gamma = 1\cdot4$).

In general, the accuracy of the method depends on the cancellation of reflected waves. Hayes & Probstein (1959) and Meyer (1960) give a discussion of this cancellation for the first problem treated above. Chester (1960) gives a discussion for the second problem. Further discussions of the limitations and accuracy of the shock-expansion method based on comparisons with numerical calculations are given by Eggers, Syvertson & Kraus (1953).

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